

Canonical and Lie-algebraic twist deformations of κ -Poincare and contractions to κ -Galilei algebras

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Abstract

We propose canonical and Lie-algebraic twist deformations of κ -deformed Poincare Hopf algebra which leads to the generalized κ -Minkowski space-time relations. The corresponding deformed κ -Poincare quantum groups are also calculated. Finally, we perform the nonrelativistic contraction limit to the corresponding twisted Galilean algebras and dual Galilean quantum groups.

1 Introduction

Recently, it has been suggested that the classical Poincare invariance should be treated as an approximate symmetry in ultra-high energy regime and the relativistic space-time symmetries on Planck scale is deformed [1]-[4]. Besides, there are also arguments based on quantum gravity [5], [6] and string theory models [7], [8] which suggest that space-time at Planck length is quantum, i.e. it should be noncommutative. The simplest choice of the noncommutative space-time is the following

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} + i\theta_{\mu\nu}^\rho \hat{x}_\rho \quad ; \quad \theta_{\mu\nu}, \theta_{\mu\nu}^\rho = \text{const} . \quad (1)$$

The first, simplest kind of noncommutativity ($\theta_{\mu\nu} \neq 0$, $\theta_{\mu\nu}^\rho = 0$ in formula (1)) was investigated in the Hopf-algebraic framework in [9]-[14]. It corresponds to the well-known canonical (soft) deformation of Poincare Hopf algebra obtained by twist procedure [15]. The second type of space-time deformation ($\theta_{\mu\nu} = 0$, $\theta_{\mu\nu}^\rho \neq 0$) is directly associated with another modification of classical relativistic symmetries - the κ -deformed Poincare Hopf algebra [16], [17], which is an example of the Lie-algebraic kind of space-time noncommutativity.

In almost all considerations both modifications of Minkowski space - Lie- and soft-type - are considered separately. Here we ask about such a deformation of relativistic space-time symmetry, when both noncommutativities will appear together, i.e. for which in the formula (1) both coefficients $\theta_{\mu\nu}$ and $\theta_{\mu\nu}^\rho$ are different from zero.

The results of Zakrzewski's ([18], [19]) indicate how to look for such a generalized Hopf Poincare structure. The classical r-matrix related to such a modification of space-time symmetries should be a sum of r-matrices for κ -Poincare group and the one describing canonical twist. Besides, this extended r-operator should solve the modified Yang-Baxter equation the same as in the case of κ -deformed Poincare symmetry. In this way, one can see that the explicit form of a proper twist factor allows us to derive a deformation of new quantum group - canonically twisted κ -Poincare Hopf algebra. Moreover, its dual partner can be calculated by a canonical quantization scheme of corresponding extended Poisson-Lie structure [20].

It should be mentioned, that the above algorithm can be generalized to two other twist deformations of κ -Poincare algebra - Lie-type [21] (see also [22]) and quadratic-one [21]. First of them leads to a Lie-algebraic noncommutativity of Minkowski space, and it introduces in natural way a second (apart of κ) mass-like parameter $\hat{\kappa}$. In the case of quadratic extension of κ -Poincare algebra the deformation parameter is dimensionless.

In this article we consider both the canonical and Lie-algebraic twist deformations of κ -Poincare symmetry. In second Section we recall necessary facts concerning the κ -deformed Poincare algebra and its dual quantum group. The canonical and Lie-algebraic deformations of κ -Poincare algebras and κ -Minkowski space-times are presented in Section 3 and 4, respectively. In Section 5 we find canonically and Lie-algebraically deformed κ -Poincare dual groups. Finally, the nonrelativistic contraction limits ([23]-[25]) to the twisted Galilean algebras and dual quantum groups [26], [23] are performed in Section 6. The results are briefly discussed and summarized in the last Section.

2 κ -Poincare deformation - short review

2.1 κ -deformed Poincare algebra

The κ -deformed Poincare algebra $\mathcal{U}_\kappa(\mathcal{P})$ is the associative and coassociative Hopf structure with generators $M_{\mu\nu}$ and P_μ satisfying the following relations [27] ($\eta_{\mu\nu} = (-, +, +, +)$)

$$[M^{\mu\nu}, M^{\lambda\sigma}] = i(\eta^{\mu\sigma} M_{\nu\lambda} - \eta^{\nu\sigma} M_{\mu\lambda} + \eta^{\nu\lambda} M_{\mu\sigma} - \eta^{\mu\lambda} M_{\nu\sigma}) , \quad (2)$$

$$[M^{ij}, P_k] = i(\delta^j_k P_i - \delta^i_k P_j) , \quad (3)$$

$$[M^{i0}, P_j] = i\delta^i_j \left[\frac{\kappa}{2} \left(1 - e^{-\frac{2P_0}{\kappa}} \right) + \frac{1}{2\kappa} \vec{P}^2 \right] - \frac{i}{\kappa} P^i P_j , \quad (4)$$

$$[M^{ij}, P_0] = 0 , \quad [M^{i0}, P_0] = iP_i , \quad [P_\mu, P_\nu] = 0 , \quad (5)$$

with the coproducts, antipodes and counits defined by

$$\Delta_\kappa(M^{ij}) = M^{ij} \otimes 1 + 1 \otimes M^{ij} , \quad (6)$$

$$\Delta_\kappa(M^{i0}) = M^{i0} \otimes e^{-\frac{P_0}{\kappa}} + 1 \otimes M^{i0} - \frac{1}{\kappa} M^{ij} \otimes P_j , \quad (7)$$

$$\Delta_\kappa(P_0) = P_0 \otimes 1 + 1 \otimes P_0 , \quad \Delta_\kappa(P_i) = P_i \otimes e^{-\frac{P_0}{\kappa}} + 1 \otimes P_i , \quad (8)$$

$$S_\kappa(M^{ij}) = -M^{ij} , \quad S_\kappa(M^{i0}) = -\left(M^{i0} + \frac{1}{\kappa} M^{ij} P_j \right) e^{\frac{P_0}{\kappa}} , \quad (9)$$

$$S_\kappa(P_i) = -P_i e^{\frac{P_0}{\kappa}} , \quad S_\kappa(P_0) = -P_0 , \quad \epsilon(P_\mu) = \epsilon(M^{\mu\nu}) = 0 . \quad (10)$$

The κ -deformed mass Casimir looks as follows

$$C_\kappa = \left(2\kappa \sinh \left(\frac{P_0}{\kappa} \right) \right)^2 - \vec{P}^2 e^{\frac{P_0}{\kappa}} . \quad (11)$$

We see that in $\mathcal{U}_\kappa(\mathcal{P})$ one can distinguish the following two Hopf subalgebras: non-deformed $O(3)$ -rotation algebra and Abelian fourmomentum algebra. For $\kappa \rightarrow \infty$ the deformation disappears and we get the classical Poincare Hopf algebra $\mathcal{U}_0(\mathcal{P})$.

It is well-known that the classical r-matrix corresponding to the above Hopf structure has the form [18], [28], [29]

$$r_\kappa = \frac{1}{\kappa} M_{0\mu} \wedge P^\mu = r_\kappa^{\mu\nu;\alpha} M_{\mu\nu} \wedge P_\alpha \quad ; \quad r_\kappa^{\mu\nu;\alpha} = \frac{1}{2\kappa} (\delta^\mu_0 \eta^{\nu\alpha} - \delta^\nu_0 \eta^{\mu\alpha}) , \quad (12)$$

with $a \wedge b = a \otimes b - b \otimes a$. One can check that the matrix (12) with itself satisfies a modified Yang-Baxter equation (MYBE)

$$[[r_\kappa, r_\kappa]] := [r_{\kappa 12}, r_{\kappa 13} + r_{\kappa 23}] + [r_{\kappa 13}, r_{\kappa 23}] = \frac{1}{\kappa^2} M_{\mu\nu} \wedge P^\mu \wedge P^\nu , \quad (13)$$

where used in the above formula symbol $[[\cdot, \cdot]]$ denotes Schouten bracket while $r_{\kappa 12} = \frac{1}{\kappa} M_{i0} \wedge P_i \wedge 1$, $r_{\kappa 13} = \frac{1}{\kappa} M_{i0} \wedge 1 \wedge P_i$ and $r_{\kappa 23} = \frac{1}{\kappa} 1 \wedge M_{i0} \wedge P_i$.

2.2 κ -deformed Poincare group

The classical r-matrix (12) defines Poisson-Lie structure [20]. Its standard quantization procedure leads to a dual form of the Hopf algebra (2)-(10) - the κ -deformed Poincare group \mathcal{P}_κ [28], [29]. It is defined by the following

a) algebraic relations

$$[\Lambda^\alpha_\beta, a^\rho] = -\frac{i}{\kappa} ((\Lambda^\alpha_0 - \delta^\alpha_0) \Lambda^\rho_\beta + \eta^{\alpha\rho} (\Lambda_{0\beta} - \eta_{0\beta})) , \quad (14)$$

$$[a^\rho, a^\sigma] = -\frac{i}{\kappa} (\delta^\sigma_0 a^\rho - \delta^\rho_0 a^\sigma) \quad , \quad [\Lambda^\alpha_\beta, \Lambda^\delta_\rho] = 0 , \quad (15)$$

b) coproducts

$$\Delta(\Lambda^\mu_\nu) = \Lambda^\mu_\alpha \otimes \Lambda^\alpha_\nu \quad , \quad \Delta(a^\mu) = \Lambda^\mu_\nu \otimes a^\nu + a^\mu \otimes 1 , \quad (16)$$

c) antipodes and counits

$$S(\Lambda^\mu_\nu) = \Lambda^\mu_\nu \quad , \quad S(a^\mu) = -\Lambda^\mu_\nu a^\nu \quad , \quad \epsilon(\Lambda^\mu_\nu) = \delta^\mu_\nu \quad , \quad \epsilon(a^\mu) = 0 . \quad (17)$$

The used above generators Λ^μ_ν are dual to $M^{\mu\nu}$ - Lorentz rotation generators

$$\langle \Lambda^\mu_\nu, M^{\alpha\beta} \rangle = (\eta^{\alpha\mu} \delta^\beta_\nu - \eta^{\beta\mu} \delta^\alpha_\nu) , \quad (18)$$

while a^μ are dual to P_μ (translations)

$$\langle a^\mu, P_\nu \rangle = \delta^\mu_\nu . \quad (19)$$

It should be noted that the relations (16)-(17) remain undeformed as for the classical Poincare group \mathcal{P} .

2.3 κ -deformed Minkowski space

It is well-known (see e.g. [30]) that the deformed Minkowski space can be introduced as the quantum representation space (a Hopf module) for quantum Poincare algebra, equipped with a proper defined \star -multiplication of two arbitrary function. Such a \star -product should be consistent with the action of deformed symmetry generators satisfying suitably deformed Leibnitz (coproduct) rules. In the case of κ -deformation the \star_κ -multiplication looks as follows (see [31], [32] and references therein)

$$f(x) \star_\kappa g(x) = \omega(\mathcal{O}_\kappa(x_\mu, \partial^\mu)(f(x) \otimes g(x))) , \quad (20)$$

where $\omega(f(x) \otimes g(x)) = f(x)g(x)$ and the \star_κ -differential operator is given by

$$\mathcal{O}_\kappa(x_\mu, \partial^\mu) := \exp(ix_\mu \gamma^\mu(\partial^\mu)) , \quad (21)$$

with

$$\gamma^\mu(\partial^\nu) := c^\mu_{\rho\tau} \partial^\rho \otimes \partial^\tau + \frac{1}{12} c^\mu_{\rho\tau} c^\rho_{\lambda\nu} (\partial^\tau \partial^\lambda \otimes \partial^\nu + \partial^\nu \otimes \partial^\tau \partial^\lambda) + \dots ; \quad (22)$$

$$c^i_{0i} = -c^i_{i0} = -\frac{1}{2\kappa} \quad \text{other} \quad c^\mu_{\rho\tau} = 0 . \quad (23)$$

Using the formula (20) in the case $f(x) = x_\mu$, $g(x) = x_\nu$ we see that the κ -deformed Minkowski space-time takes the form

$$[x_i, x_0]_{\star_\kappa} = x_i \star_\kappa x_0 - x_0 \star_\kappa x_i = \frac{i}{\kappa} x_i \quad , \quad [x_i, x_j]_{\star_\kappa} = 0 , \quad (24)$$

and in the $\kappa \rightarrow \infty$ limit it becomes classical.

3 Canonical twist deformation of κ -Poincare algebra

3.1 Extended classical r-matrix

Let us consider the following extension of classical r-matrix (12)

$$r = r_\kappa + r_{\hat{\kappa}} + r_\xi , \quad (25)$$

with

$$r_{\hat{\kappa}} = \frac{1}{2\hat{\kappa}} M_{12} \wedge P_0 , \quad (26)$$

and

$$r_\xi = \frac{\xi}{2} P_3 \wedge P_0 , \quad (27)$$

where the formulas (26) and (27) describe Lie-algebraic and canonical twist deformations of κ -Poincare algebra, respectively. Due to the commutation relations $[P_\mu, P_\nu] = [M_{12}, P_3] = 0$ (see (3) and (5)) we can see that both matrices $r_{\hat{\kappa}}$ and r_ξ satisfy the classical Yang-Baxter equation (CYBE)

$$[[r_{\hat{\kappa}}, r_{\hat{\kappa}}]] = [[r_\xi, r_\xi]] = 0 ; \quad (28)$$

the mixed Schouten brackets vanish as well

$$[[r_{\hat{\kappa}}, r_\xi]] = [[r_\xi, r_{\hat{\kappa}}]] = 0 . \quad (29)$$

By explicit calculation one can check that

$$[[r_\kappa, r.]] = [[r., r_\kappa]] = 0 \quad ; \quad r. = r_{\hat{\kappa}}, r_\xi , \quad (30)$$

which together with the formulas (28) and (29) means that the extended r-matrix (25) satisfies the modified Yang-Baxter equation (13)

$$[[r, r]] = \frac{1}{\kappa^2} M_{\mu\nu} \wedge P^\mu \wedge P^\nu . \quad (31)$$

3.2 Canonical deformation of κ -Poincare algebra

In accordance with (30) one can consider the canonical ($r = r_\kappa + r_\xi$) deformation of enveloping κ -Poincare algebra $\mathcal{U}_\kappa(\mathcal{P})$. As already mentioned in Introduction we can get such a modification of space-time relativistic symmetry by a proper (κ -deformed) twisting procedure.

First of all, let us introduce an element $\mathcal{F}_\xi \in \mathcal{U}_\kappa(\mathcal{P}) \otimes \mathcal{U}_\kappa(\mathcal{P})$ with the following linear term in series expansion with respect to the deformation parameter ξ

$$\mathcal{F}_\xi = 1 + i r_\xi^{(1)} \otimes r_\xi^{(2)} + \dots ; \quad r_\xi = r_\xi^{(1)} \otimes r_\xi^{(2)} . \quad (32)$$

Next, we define Drinfeld twist factor as the function (32) satisfying so-called κ -deformed cocycle condition [33]

$$\mathcal{F}_{\xi 12} \cdot (\Delta_\kappa \otimes 1) \mathcal{F}_\xi = \mathcal{F}_{\xi 23} \cdot (1 \otimes \Delta_\kappa) \mathcal{F}_\xi , \quad (33)$$

and the normalization condition

$$(\epsilon \otimes 1) \mathcal{F}_\xi = (1 \otimes \epsilon) \mathcal{F}_\xi = 1 , \quad (34)$$

with $\mathcal{F}_{\xi 12} = \mathcal{F}_\xi \otimes 1$ and $\mathcal{F}_{\xi 23} = 1 \otimes \mathcal{F}_\xi$. The solution of above equations has been found in [19] and it looks as follows

$$\mathcal{F}_{\xi, \kappa} = \exp \left(i \kappa \frac{\xi}{2} P_3 \otimes \left(e^{-\frac{P_0}{\kappa}} - 1 \right) \right) . \quad (35)$$

One can easily see that in the limit $\xi \rightarrow 0$ factor $\mathcal{F}_{\xi, \kappa}$ goes to the unit operator

$$\lim_{\xi \rightarrow 0} \mathcal{F}_{\xi, \kappa} = 1 , \quad (36)$$

while in the case $\kappa \rightarrow \infty$ we get a standard canonical-twist element for the classical Poincare Hopf algebra $\mathcal{U}_0(\mathcal{P})$

$$\mathcal{F}_{\xi, \infty} = e^{-i \frac{\xi}{2} P_3 \otimes P_0} . \quad (37)$$

It is well-known that twist $\mathcal{F}_{\xi, \kappa}$ does not modify the algebraic part of κ -Poincare algebra (2)-(5) and counits, but it changes the coproducts (6)-(8) and antipodes (9), (10) according to

$$\Delta_{\mathcal{F}_{\xi, \kappa}}(a) = \mathcal{F}_{\xi, \kappa} \Delta_\kappa(a) \mathcal{F}_{\xi, \kappa}^{-1} , \quad (38)$$

$$S_{\mathcal{F}_{\xi, \kappa}}(a) = u(\kappa, \xi) S_\kappa(a) u^{-1}(\kappa, \xi) , \quad (39)$$

where $u(\kappa, \xi) = \sum f_{(1)} S_\kappa(f_{(2)})$, and where we use Sweedler's notation $\mathcal{F}_{\xi, \kappa} = \sum f_{(1)} \otimes f_{(2)}$. Hence, using the formula

$$u(\kappa, \xi) = \exp (i \kappa \xi P_3 (\exp(P_0/\kappa) - 1)) , \quad (40)$$

we obtain

$$\Delta_{\mathcal{F}_{\xi,\kappa}}(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \quad , \quad \Delta_{\mathcal{F}_{\xi,\kappa}}(P_i) = P_i \otimes e^{-\frac{P_0}{\kappa}} + 1 \otimes P_i \quad , \quad (41)$$

$$\Delta_{\mathcal{F}_{\xi,\kappa}}(M^{ij}) = \Delta_{\kappa}(M^{ij}) + \kappa \frac{\xi}{2} (\delta^j_3 P_i - \delta^i_3 P_j) \otimes \left(e^{-\frac{P_0}{\kappa}} - 1 \right) \quad , \quad (42)$$

$$\Delta_{\mathcal{F}_{\xi,\kappa}}(M^{i0}) = \Delta_{\kappa}(M^{i0}) - \frac{\xi}{2} P_3 \otimes P_i e^{-\frac{P_0}{\kappa}} + \quad (43)$$

$$+ \frac{\xi}{2} \left(\delta^i_3 \left[\frac{\kappa}{2} \left(1 - e^{-\frac{2P_0}{\kappa}} \right) + \frac{1}{2\kappa} \vec{P}^2 \right] + \frac{1}{\kappa} P_i P_3 \right) \otimes \quad (44)$$

$$\otimes \kappa \left(e^{-\frac{P_0}{\kappa}} - 1 \right) e^{-\frac{P_0}{\kappa}} + \quad (45)$$

$$+ \frac{\xi}{2} (\delta^j_3 P_i - \delta^i_3 P_j) \otimes P_j \left(e^{-\frac{P_0}{\kappa}} - 1 \right) \quad , \quad (46)$$

and

$$S_{\mathcal{F}_{\xi,\kappa}}(P_0) = S_{\kappa}(P_0) = -P_0 \quad , \quad S_{\mathcal{F}_{\xi,\kappa}}(P_i) = S_{\kappa}(P_i) = -P_i e^{\frac{P_0}{\kappa}} \quad , \quad (47)$$

$$S_{\mathcal{F}_{\xi,\kappa}}(M^{ij}) = S_{\kappa}(M^{ij}) - \kappa \xi (\delta^j_3 P_i - \delta^i_3 P_j) \cdot (\exp(P_0/\kappa) - 1) \quad , \quad (48)$$

$$S_{\mathcal{F}_{\xi,\kappa}}(M^{i0}) = S_{\kappa}(M^{i0}) - \xi (\delta^j_3 P_i - \delta^i_3 P_j) P_j \cdot e^{\frac{P_0}{\kappa}} \quad . \quad (49)$$

$$\cdot (\exp(P_0/\kappa) - 1) - \xi P_3 P_i e^{\frac{2P_0}{\kappa}} + \quad (50)$$

$$- \kappa \xi \left(\delta^i_3 \left[\frac{\kappa}{2} \left(1 - e^{-\frac{2P_0}{\kappa}} \right) + \frac{1}{2\kappa} \vec{P}^2 \right] - \frac{1}{\kappa} P_i P_3 \right) \cdot \quad (51)$$

$$\cdot e^{\frac{P_0}{\kappa}} \cdot (\exp(P_0/\kappa) - 1) \quad . \quad (52)$$

The algebraic relations (2)-(5) together with coproducts (41)-(46), antipodes (47)-(52) and classical counits (10) define the canonical twist deformation of κ -Poincare algebra $\mathcal{U}_{\xi,\kappa}(\mathcal{P})$. We see, that for $\xi \rightarrow 0$ one gets the κ -Poincare algebra $\mathcal{U}_{\kappa}(\mathcal{P})$, which is in accordance with the formula (36), i.e. there is no twist transformation in such a case. For parameter $\kappa \rightarrow \infty$, the algebra $\mathcal{U}_{\xi,\kappa}(\mathcal{P})$ passes into well-known $(\theta^{\mu\nu})$ -Poincare Hopf structure [10], and this time, it agrees with the form of twist factor (37).

3.3 Canonical extension of κ -Minkowski space

Let us now find a noncommutative Minkowski space corresponding to the canonical deformation of κ -Poincare. As it was mentioned in the first section, our space-time can be defined as a quantum representation space for the extended quantum Poincare algebra

$\mathcal{U}_{\xi,\kappa}(\mathcal{P})$, equipped with a proper deformed \star -multiplication. We define our \star -product for arbitrary two functions depending on space-time coordinates as follows

$$f(x) \star_{\kappa,\xi} g(x) = \omega(\mathcal{O}_{\kappa,\xi}(x_\mu, \partial^\mu)(f(x) \otimes g(x))) , \quad (53)$$

where the \star -operator $\mathcal{O}_{\kappa,\xi}(x_\mu, \partial^\mu)$ is given by the superposition of two \star -operators: for the κ -deformed r -matrix r_κ (see (21)), and for the canonical deformed matrix r_ξ (see twist factor (35)) [30]

$$\mathcal{O}_\xi(x_\mu, \partial^\mu) := \mathcal{F}_{\xi,\kappa}^{-1}(x_\mu, \partial^\mu) = \exp\left(-i\kappa\frac{\xi}{2}\partial^3 \otimes \left(e^{-\frac{\partial^0}{\kappa}} - 1\right)\right) . \quad (54)$$

Consequently, our operator takes the form

$$\mathcal{O}_{\kappa,\xi}(x_\mu, \partial^\mu) := \mathcal{O}_\xi(x_\mu, \partial^\mu) \circ \mathcal{O}_\kappa(x_\mu, \partial^\mu) , \quad (55)$$

and we obtain the following commutation relations

$$[x_i, x_0]_{\star_{\kappa,\xi}} = \frac{i}{\kappa}x_i + i\frac{\xi}{2}\delta_i^3 , \quad [x_i, x_j]_{\star_{\kappa,\xi}} = 0 . \quad (56)$$

The relations (56) define the canonically extended κ -Minkowski space-time $\mathcal{M}_{\kappa,\xi}$. We see that the soft deformation of κ -Poincare algebra introduces two kinds of noncommutativity: Lie-type associated with parameter κ , and canonical type - corresponding to parameter ξ . Of course, for $\xi \rightarrow 0$ one gets the κ -deformed Minkowski space-time \mathcal{M}_κ , while in the $\kappa \rightarrow \infty$ limit we obtain well-known $\theta^{\mu\nu}$ -deformed Minkowski space \mathcal{M}_θ (see e.g. [10]).

4 Lie-algebraic twist deformation of κ -Poincare algebra

4.1 Deformation of algebra

In the case of Lie-algebraic deformation ($r = r_\kappa + r_{\hat{\kappa}}$) the twist factor has been found in [19]. Here we consider its antisymmetric form

$$\mathcal{F}_{\hat{\kappa},\kappa} = \exp\left(\frac{i}{2\hat{\kappa}}M_{12} \wedge P_0\right) . \quad (57)$$

By tedious calculation we get the following coproduct of deformed κ -Poincare algebra $\mathcal{U}_{\hat{\kappa},\kappa}(\mathcal{P})$

$$\Delta_{\mathcal{F}_{\hat{\kappa},\kappa}}(P_0) = P_0 \otimes 1 + 1 \otimes P_0 , \quad \Delta_{\mathcal{F}_{\hat{\kappa},\kappa}}(P_3) = P_3 \otimes e^{-\frac{P_0}{\kappa}} + 1 \otimes P_3 , \quad (58)$$

$$\Delta_{\mathcal{F}_{\hat{\kappa},\kappa}}(P_1) = \Delta_\kappa(P_1) - \sin\left(\frac{P_0}{2\hat{\kappa}}\right) \otimes P_2 + P_2 \otimes \sin\left(\frac{P_0}{2\hat{\kappa}}\right) e^{-\frac{P_0}{\kappa}} + \quad (59)$$

$$- \left[\cos \left(\frac{P_0}{2\hat{\kappa}} \right) - 1 \right] \otimes P_1 - P_1 \otimes \left[\cos \left(\frac{P_0}{2\hat{\kappa}} \right) - 1 \right] e^{-\frac{P_0}{\kappa}}, \quad (60)$$

$$\Delta_{\mathcal{F}_{\hat{\kappa},\kappa}}(P_2) = \Delta_{\kappa}(P_2) + \sin \left(\frac{P_0}{2\hat{\kappa}} \right) \otimes P_1 - P_1 \otimes \sin \left(\frac{P_0}{2\hat{\kappa}} \right) e^{-\frac{P_0}{\kappa}} + \quad (61)$$

$$- \left[\cos \left(\frac{P_0}{2\hat{\kappa}} \right) - 1 \right] \otimes P_2 - P_2 \otimes \left[\cos \left(\frac{P_0}{2\hat{\kappa}} \right) - 1 \right] e^{-\frac{P_0}{\kappa}}, \quad (62)$$

$$\Delta_{\mathcal{F}_{\hat{\kappa},\kappa}}(M^{ij}) = \Delta_{\kappa}(M^{ij}) - i[M^{ij}, M^{12}] \wedge \sin \left(\frac{P_0}{2\hat{\kappa}} \right) + \quad (63)$$

$$- [[M^{ij}, M^{12}], M^{12}] \perp \left[\cos \left(\frac{P_0}{2\hat{\kappa}} \right) - 1 \right], \quad (64)$$

$$\begin{aligned} \Delta_{\mathcal{F}_{\hat{\kappa},\kappa}}(M^{i0}) &= \\ &= \Delta_{\kappa}(M^{i0}) - \frac{1}{2\hat{\kappa}} M^{12} \otimes P_i + \frac{1}{2\hat{\kappa}} P_i \otimes M^{12} e^{-\frac{P_0}{\kappa}} - i[M^{i0}, M^{12}] \otimes \sin \left(\frac{P_0}{2\hat{\kappa}} \right) e^{-\frac{P_0}{\kappa}} \\ &+ i \sin \left(\frac{P_0}{2\hat{\kappa}} \right) \otimes [M^{i0}, M^{12}] - [[M^{i0}, M^{12}], M^{12}] \otimes \left[\cos \left(\frac{P_0}{2\hat{\kappa}} \right) - 1 \right] e^{-\frac{P_0}{\kappa}} \\ &- \left[\cos \left(\frac{P_0}{2\hat{\kappa}} \right) - 1 \right] \otimes [[M^{i0}, M^{12}], M^{12}] - \frac{1}{2\hat{\kappa}} M^{12} \sin \left(\frac{P_0}{2\hat{\kappa}} \right) \otimes (\delta^{1i} P_2 - \delta^{2i} P_1) \\ &- \frac{1}{2\hat{\kappa}} (\delta^{1i} P_2 - \delta^{2i} P_1) \otimes M^{12} \sin \left(\frac{P_0}{2\hat{\kappa}} \right) e^{-\frac{P_0}{\kappa}} \\ &+ \frac{1}{2\hat{\kappa}} (\delta^{1i} P_1 + \delta^{2i} P_2) \otimes M^{12} \left[\cos \left(\frac{P_0}{2\hat{\kappa}} \right) - 1 \right] e^{-\frac{P_0}{\kappa}} \end{aligned} \quad (65)$$

$$\begin{aligned} &- \frac{1}{2\hat{\kappa}} M^{12} \left[\cos \left(\frac{P_0}{2\hat{\kappa}} \right) - 1 \right] \otimes (\delta^{1i} P_1 + \delta^{2i} P_2) + \frac{i}{\kappa} [M^{ij}, M^{12}] \otimes \sin \left(\frac{P_0}{2\hat{\kappa}} \right) P_j \\ &+ \frac{1}{\kappa} [[M^{ij}, M^{12}], M^{12}] \otimes \left[\cos \left(\frac{P_0}{2\hat{\kappa}} \right) - 1 \right] P_j + \frac{1}{\kappa} \sin \left(\frac{P_0}{2\hat{\kappa}} \right) M^{ij} \otimes (\delta^{1j} P_2 - \delta^{2j} P_1) \\ &+ \frac{1}{\kappa} \left[\cos \left(\frac{P_0}{2\hat{\kappa}} \right) - 1 \right] M^{ij} \otimes (\delta^{1j} P_1 + \delta^{2j} P_2). \end{aligned}$$

The algebraic sector as well as the antipodes remain κ -deformed, i.e. $S_{\mathcal{F}_{\hat{\kappa},\kappa}}(a) = S_{\kappa}(a)$ (see (9), (10)).

4.2 Two-parameter extension of κ -Minkowski space

For Lie-algebraic deformation we define the \star -operator as follows

$$\mathcal{O}_{\kappa,\hat{\kappa}}(x_{\mu}, \partial^{\mu}) := \mathcal{O}_{\hat{\kappa}}(x_{\mu}, \partial^{\mu}) \circ \mathcal{O}_{\kappa}(x_{\mu}, \partial^{\mu}); \quad (66)$$

$$\mathcal{O}_{\hat{\kappa}}(x_\mu, \partial^\mu) := \mathcal{F}_{\hat{\kappa}, \kappa}^{-1}(x_\mu, \partial^\mu) = \exp \left(-\frac{i}{2\hat{\kappa}}(x_1\partial^2 - x_2\partial^1) \wedge \partial^0 \right) , \quad (67)$$

and our $(\kappa, \hat{\kappa})$ -deformed Minkowski space takes the form

$$[x_i, x_0]_{\star_{\kappa, \hat{\kappa}}} = \frac{i}{\kappa}x_i + \frac{i}{\hat{\kappa}}(\delta_i^1 x_2 - \delta_i^2 x_1) \quad , \quad [x_i, x_j]_{\star_{\kappa, \hat{\kappa}}} = 0 . \quad (68)$$

The relations (68) define the Lie-algebraic extension of κ -Minkowski space-time $\mathcal{M}_{\kappa, \hat{\kappa}}$. We see that above deformation of κ -Poincare algebra introduces Lie-algebraic type of space-time noncommutativity corresponding to both parameters κ and $\hat{\kappa}$. It should be also noted that in the $\hat{\kappa} \rightarrow \infty$ limit we get the κ -deformed Minkowski space-time \mathcal{M}_κ , while for $\kappa \rightarrow \infty$ we obtain the Minkowski space for Lie-twisted Poincare algebra $\mathcal{M}_{\hat{\kappa}}$ [22].

5 Canonical and Lie-algebraic twist deformation of κ -Poincare group

In accordance with the equation (31) one can define the corresponding to the matrix (25) Poisson-Lie structure as follows [20]

$$\{f, g\} = 2r^{AB} (X_A^R f X_B^R g - X_A^L f X_B^L g) . \quad (69)$$

The symbols X_A^R, X_A^L denote the right- and left-invariant vector fields on classical Poincare group \mathcal{P} given by

$$X_L^{\alpha\beta} = \Lambda^{\mu\alpha} \frac{\partial}{\partial \Lambda_\beta^\mu} - \Lambda^{\mu\beta} \frac{\partial}{\partial \Lambda_\alpha^\mu} \quad , \quad X_L^\alpha = \Lambda^{\mu\alpha} \frac{\partial}{\partial a^\mu} , \quad (70)$$

$$X_R^{\alpha\beta} = \Lambda^\beta_\nu \frac{\partial}{\partial \Lambda^{\alpha\nu}} - \Lambda^\alpha_\nu \frac{\partial}{\partial \Lambda^{\beta\nu}} + a^\beta \frac{\partial}{\partial a_\alpha} - a^\alpha \frac{\partial}{\partial a_\beta} \quad , \quad X_R^\alpha = \frac{\partial}{\partial a_\alpha} . \quad (71)$$

If we calculate the Poisson brackets (69) with use of the formulas (12), (26) and (27), in a first step, and if we perform its standard quantization by replacing $\{\cdot, \cdot\} \rightarrow \frac{1}{i}[\cdot, \cdot]$, as a second step, then we obtain the following set of commutation relations

$$[\Lambda^\alpha_\beta, a^\rho] = -\frac{i}{\kappa}((\Lambda^\alpha_0 - \delta^\alpha_0)\Lambda^\rho_\beta + \eta^{\alpha\rho}(\Lambda_{0\beta} - \eta_{0\beta})) + \quad (72)$$

$$+ \frac{1}{\hat{\kappa}}(\Lambda^\rho_0(\eta_{2\beta}\Lambda^\alpha_1 - \eta_{1\beta}\Lambda^\alpha_2) + \delta^\rho_0(\delta^\alpha_2\Lambda_{1\beta} - \delta^\alpha_1\Lambda_{2\beta})) , \quad (73)$$

$$[a^\rho, a^\sigma] = -\frac{i}{\kappa}(\delta^\sigma_0 a^\rho - \delta^\rho_0 a^\sigma) + \frac{i}{\hat{\kappa}}\delta^\sigma_0(\delta^\rho_2 a^1 - \delta^\rho_1 a^2) + \quad (74)$$

$$+ \frac{i}{\hat{\kappa}}\delta^\rho_0(\delta^\sigma_1 a^2 - \delta^\sigma_2 a^1) + i\frac{\xi}{2}(\delta^\rho_3 \delta^\sigma_0 - \delta^\rho_0 \delta^\sigma_3) + \quad (75)$$

$$+ i\frac{\xi}{2}(\Lambda^\rho_0 \Lambda^\sigma_3 - \Lambda^\rho_3 \Lambda^\sigma_0) \quad , \quad [\Lambda^\alpha_\beta, \Lambda^\delta_\rho] = 0 . \quad (76)$$

Next, if we define the $*$ -operation in such a way that Λ^μ_ν and a^μ are selfadjoint elements, we see that the above relations together with coproducts (16), counits and antipodes (17) give a Hopf $*$ -algebra - the $(\hat{\kappa}, \xi)$ -deformed κ -Poincare group $\mathcal{P}_{\kappa, \hat{\kappa}, \xi}$. In such a way for $\hat{\kappa} = \infty$ we get dual group to the canonically deformed algebra $\mathcal{U}_{\kappa, \xi}(\mathcal{P})$, while for $\xi = 0$ we obtain dual partner for $\mathcal{U}_{\kappa, \hat{\kappa}}(\mathcal{P})$.

It should be also noted that for $\kappa \rightarrow \infty$, $\hat{\kappa} \rightarrow \infty$ and $\xi \rightarrow 0$ we obtain the classical (undeformed) Poincare group \mathcal{P} . For $\kappa \rightarrow \infty$ and $\xi \rightarrow 0$ we get the Lie-algebraically twisted classical Poincare group [21], while in the case $\kappa \rightarrow \infty$ and $\hat{\kappa} \rightarrow \infty$ we obtain the canonical deformation of classical Poincare Hopf algebra \mathcal{P}_ξ (see [9]).

6 Contractions to twisted κ -Galilei algebras and κ -Galilei groups

In this section we calculate the nonrelativistic contractions of Hopf structures derived in previous sections, i.e. we find their nonrelativistic counterparts - the canonical and Lie-algebraic twist deformations of κ -Galilei algebra.

6.1 Canonical deformation of κ -Galilei algebra

Let us introduce the following standard redefinition of Poincaré generators [34] (see also [35])

$$P_0 = \frac{\Pi_0}{c} \quad , \quad P_i = \Pi_i \quad , \quad M_{ij} = K_{ij} \quad , \quad M_{i0} = cV_i \quad , \quad (77)$$

where parameter c describes the light velocity. We start with canonical twisted algebra $\mathcal{U}_{\xi, \kappa}(\mathcal{P})$, i.e. we introduce two parameters $\bar{\kappa}$ and $\bar{\xi}$ such that $\kappa = \bar{\kappa}/c$ and $\xi = \bar{\xi}c$. Next, one performs the contraction limit of algebraic part (2)-(5) and co-sector (41)-(46) in two steps (see e.g. [23]). Firstly, we rewrite the formulas (2)-(5) and (41)-(46) in term of the operators (77) and parameters $\bar{\kappa}$, $\bar{\xi}$. Secondly, we take the $c \rightarrow \infty$ limit, and in such a way, we get the following algebraic

$$[K^{ij}, K^{kl}] = i(\delta^{il} K^{jk} - \delta^{jl} K^{ik} + \delta^{jk} K^{il} - \delta^{ik} K^{jl}) \quad , \quad (78)$$

$$[K^{ij}, V^k] = i(\delta^{jk} V^i - \delta^{ik} V^j) \quad , \quad [K^{ij}, \Pi_k] = i(\delta^j_k \Pi_i - \delta^i_k \Pi_j) \quad , \quad (79)$$

$$[V_i, V_j] = 0 \quad , \quad [V^i, \Pi_0] = i\Pi_i \quad , \quad [\Pi_\rho, \Pi_\sigma] = 0 \quad , \quad (80)$$

$$[V^i, \Pi_j] = \delta^i_j \frac{1}{2\bar{\kappa}} \vec{\Pi}^2 - \frac{1}{\bar{\kappa}} \Pi_i \Pi_j \quad , \quad C_{\bar{\kappa}} = \vec{\Pi}^2 e^{\frac{\Pi_0}{\bar{\kappa}}} \quad , \quad (81)$$

and coalgebraic

$$\Delta_{\bar{\xi}, \bar{\kappa}}(\Pi_0) = \Pi_0 \otimes 1 + 1 \otimes \Pi_0 \quad , \quad \Delta_{\bar{\xi}, \bar{\kappa}}(\Pi_i) = \Pi_i \otimes e^{-\frac{\Pi_0}{\bar{\kappa}}} + 1 \otimes \Pi_i \quad , \quad (82)$$

$$\Delta_{\bar{\xi}, \bar{\kappa}}(K^{ij}) = \Delta_{\bar{\kappa}}(K^{ij}) + \bar{\kappa} \frac{\bar{\xi}}{2} (\delta^j_3 \Pi_i - \delta^i_3 \Pi_j) \otimes \left(e^{-\frac{\Pi_0}{\bar{\kappa}}} - 1 \right), \quad (83)$$

$$\Delta_{\bar{\xi}, \bar{\kappa}}(V^i) = \Delta_{\bar{\kappa}}(V^i) - \frac{\bar{\xi}}{2} \Pi_3 \otimes \Pi_i e^{-\frac{\Pi_0}{\bar{\kappa}}} + \quad (84)$$

$$+ \frac{\bar{\xi}}{2} \left(\delta^i_3 \frac{1}{2\bar{\kappa}} \vec{\Pi}^2 + \frac{1}{\bar{\kappa}} \Pi_i \Pi_3 \right) \otimes \bar{\kappa} \left(e^{-\frac{\Pi_0}{\bar{\kappa}}} - 1 \right) e^{-\frac{\Pi_0}{\bar{\kappa}}} \quad (85)$$

$$+ \frac{\bar{\xi}}{2} (\delta^j_3 \Pi_i - \delta^i_3 \Pi_j) \otimes \Pi_j \left(e^{-\frac{\Pi_0}{\bar{\kappa}}} - 1 \right), \quad (86)$$

sectors, where $\Delta_{\bar{\kappa}}(a) = \Delta_{\kappa}(a)$. The antipodes look as follows

$$S_{\bar{\xi}, \bar{\kappa}}(\Pi_0) = S_{\bar{\kappa}}(\Pi_0) = -\Pi_0, \quad S_{\bar{\xi}, \bar{\kappa}}(\Pi_i) = S_{\bar{\kappa}}(\Pi_i) = -\Pi_i e^{\frac{\Pi_0}{\bar{\kappa}}}, \quad (87)$$

$$S_{\bar{\xi}, \bar{\kappa}}(K^{ij}) = S_{\bar{\kappa}}(K^{ij}) - \bar{\kappa} \bar{\xi} (\delta^j_3 \Pi_i - \delta^i_3 \Pi_j) \cdot (\exp(\Pi_0/\bar{\kappa}) - 1), \quad (88)$$

$$S_{\bar{\xi}, \bar{\kappa}}(V^i) = S_{\bar{\kappa}}(V^i) - \bar{\xi} (\delta^j_3 \Pi_i - \delta^i_3 \Pi_j) \Pi_j \cdot e^{\frac{\Pi_0}{\bar{\kappa}}}. \quad (89)$$

$$\cdot (\exp(\Pi_0/\bar{\kappa}) - 1) - \bar{\xi} \Pi_3 \Pi_i e^{\frac{2\Pi_0}{\bar{\kappa}}} + \quad (90)$$

$$- \bar{\kappa} \bar{\xi} \left(\delta^i_3 \frac{1}{2\bar{\kappa}} \vec{\Pi}^2 - \frac{1}{\bar{\kappa}} \Pi_i \Pi_3 \right) \cdot e^{\frac{\Pi_0}{\bar{\kappa}}} \cdot (\exp(\Pi_0/\bar{\kappa}) - 1), \quad (91)$$

with $S_{\bar{\kappa}}(a) = S_{\kappa}(a)$. The relations (78)-(91) define the canonically twisted κ -Galilei algebra $\mathcal{U}_{\bar{\xi}, \bar{\kappa}}(\mathcal{G})$. One can see that for $\bar{\xi} \rightarrow 0$ we get the $\bar{\kappa}$ -deformed Galilei group $\mathcal{U}_{\bar{\kappa}}(\mathcal{G})$ firstly studied in [26] (see also [23]). In $\bar{\kappa} \rightarrow \infty$ limit we obtain the canonically deformed algebra $\mathcal{U}_{\bar{\xi}}(\mathcal{G})$ found in [25]. Obviously, for $\bar{\kappa} \rightarrow \infty$ and $\bar{\xi} \rightarrow 0$ one gets the undeformed Galilei quantum group $\mathcal{U}_0(\mathcal{G})$.

6.2 Lie-algebraic deformation of κ -Galilei algebra

In the case of Lie-algebraic modification of κ -Poincare algebra, we perform contraction with respect the parameters $\kappa = \bar{\kappa}/c$ and $\hat{\kappa} = \bar{\kappa}/c$. Due to the relations (58)-(65) we obtain the coproducts $\Delta_{\bar{\kappa}, \bar{\kappa}}(\Pi_\rho)$, $\Delta_{\bar{\kappa}, \bar{\kappa}}(K^{ij})$ and $\Delta_{\bar{\kappa}, \bar{\kappa}}(V^i)$ such that $\Delta_{\bar{\kappa}, \bar{\kappa}}(a) = \Delta_{\mathcal{F}_{\bar{\kappa}, \kappa}}(a)$. In this way we get the Lie-twisted Galilei algebra $\mathcal{U}_{\bar{\kappa}, \bar{\kappa}}(\mathcal{G})$, which for $\bar{\kappa} \rightarrow \infty$ passes into $\bar{\kappa}$ -deformed Galilei group $\mathcal{U}_{\bar{\kappa}}(\mathcal{G})$.

6.3 Canonical and Lie-algebraic deformation of κ -Galilei group

Finally, let us find the contraction of $(\hat{\kappa}, \xi)$ -deformed Poincare group $\mathcal{P}_{\kappa, \hat{\kappa}, \xi}$ (see (72)-(76) and (16), (17)). In this purpose we introduce the following redefinition of Λ^μ_ν , a^μ

generators [36]

$$\Lambda^0_0 = \left(1 + \frac{\bar{v}^2}{c^2}\right)^{\frac{1}{2}}, \quad \Lambda^i_0 = \frac{v^i}{c}, \quad \Lambda^0_i = \frac{v^k R^k_i}{c}, \quad (92)$$

$$\Lambda^k_i = \left(\delta^k_l + \left(\left(1 + \frac{\bar{v}^2}{c^2}\right)^{\frac{1}{2}} - 1\right) \frac{v^k v^l}{\bar{v}^2}\right) R^l_i, \quad (93)$$

$$a^i = b^i, \quad a^0 = c\tau, \quad (94)$$

where $\{R^i_j, v^i, \tau, b^i\}$ denote the generators of Galilei group. With use of the formulas (92)-(94) in the contraction limit $c \rightarrow \infty$ we get

$$[R^k_l, b^i] = -\frac{i}{\bar{\kappa}}(v^k R^i_l - \delta^{ki} v^\rho R^\rho_l) + \frac{1}{\bar{\kappa}} v^i (\delta_{2l} R^k_1 - \delta_{1l} R^k_2), \quad (95)$$

$$[R^k_l, \tau] = \frac{1}{\bar{\kappa}}((\delta_{2l} R^k_1 - \delta_{1l} R^k_2) - (\delta^k_2 R_{1l} - \delta^k_1 R_{2l})), \quad (96)$$

$$[v^i, b^j] = -\frac{i}{\bar{\kappa}}(v^i v^j - \frac{1}{2} \delta^{ij} \bar{v}^2), \quad [v^i, \tau] = -\frac{i}{\bar{\kappa}} v^i - \frac{1}{\bar{\kappa}} (\delta^i_2 v_1 - \delta^i_1 v_2), \quad (97)$$

$$[\tau, b^i] = -\frac{i}{\bar{\kappa}} b^i + \frac{i}{\bar{\kappa}} (\delta^i_2 b^1 - \delta^i_1 b^2) + i \frac{\bar{\xi}}{2} (R^i_3 + \delta^i_3), \quad (98)$$

$$[b^i, b^j] = i \frac{\bar{\xi}}{2} (v^i R^j_3 - R^i_3 v^j), \quad (99)$$

$$[R^i_j, R^k_l] = [v^i, R^k_l] = [v^i, v^j] = 0. \quad (100)$$

The coproducts remain undeformed

$$\Delta(R^i_j) = R^i_k \otimes R^k_j, \quad \Delta(v^i) = R^i_j \otimes v^j + v^i \otimes 1, \quad (101)$$

$$\Delta(\tau) = \tau \otimes 1 + 1 \otimes \tau, \quad \Delta(b^i) = R^i_j \otimes b^j + v^i \otimes \tau + b^i \otimes 1. \quad (102)$$

The relations (95)-(102) with classical antipodes and counits define the $(\hat{\kappa}, \xi)$ -deformed Galilei group $\mathcal{G}_{\kappa, \hat{\kappa}, \xi}$. As in the case of relativistic symmetries, for $\hat{\kappa} = \infty$ we get dual group to the Galilei algebra $\mathcal{U}_{\xi, \bar{\kappa}}(\mathcal{G})$, while for $\xi = 0$ we obtain dual partner for the algebra $\mathcal{U}_{\bar{\kappa}, \bar{\kappa}}(\mathcal{G})$.

Finally, one should also notice that in the $\bar{\kappa} \rightarrow \infty$ and $\bar{\xi} \rightarrow 0$ limits we get the well-known $\bar{\kappa}$ -deformed Galilei group $\mathcal{G}_{\bar{\kappa}}$ (see [36]), while for $\bar{\kappa} \rightarrow \infty$ and $\bar{\xi} \rightarrow 0$ or $\bar{\kappa} \rightarrow \infty$, we obtain the quantum Galilei groups recovered in [37].

7 Final remarks

In this article we introduced two twist extensions of κ -Minkowski spaces corresponding to soft and Lie-algebraic type of noncommutativity (see (56) and (68)). For such modified

space-times we find their quantum Poincare algebras and corresponding dual quantum groups. The nonrelativistic contractions are performed as well.

As it was mentioned in Introduction the Lie-algebraic twist introduces in natural way a second mass-like parameter of deformation. Consequently, in such a way, one can obtain a "modification" of so-called Doubly Special Relativity ([38]-[40]) with one fundamental mass parameter, by introducing a second observer-independent mass-like scale.

It should be also noted that this paper is only a starting point for a further investigation. For example, it is interesting to ask about the noncommutative field theory given on such generalized quantum Minkowski space-times. In particular, its formulation requires the construction of a proper differential calculus, a proper star product of fields, and a suitable deformation of statistics for creation/annihilation operators (see e.g. [41]-[44]). The above problems are now under considerations and they are postponed for further investigation.

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